SOLITON SOLUTIONS OF THE TODA HIERARCHY ON QUASI-PERIODIC BACKGROUNDS REVISITED

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Abstract. We investigate soliton solutions of the Toda hierarchy on a quasi-periodic finite-gap background by means of the double commutation method and the inverse scattering transform. In particular, we compute the phase shift caused by a soliton on a quasi-periodic finite-gap background. Furthermore, we consider short range perturbations via scattering theory. We give a full description of the effect of the double commutation method on the scattering data and establish the inverse scattering transform in this setting.

1. Introduction

Solitons on a (quasi-)periodic background have a long tradition and are used to model localized excitements on a phonon, lattice, or magnetic field background (see, e.g., [5], [11], [13], [15], [16], [17], [18] and the references therein). Of course periodic solutions, as well as solitons travelling on a periodic background, are well understood. Nevertheless there are still several open questions.

One of them is the stability of (quasi-)periodic solutions. For the constant solution it is a classical result, that a small initial perturbation asymptotically splits in a number of stable solitons. For a (quasi-)periodic background this cannot be the case. In fact, associated with every soliton there is a phase shift (which will be explicitly computed in Section 4) and the phase shifts of all solitons will not add up to zero in general. Hence there must be something which makes up for this phase shift. Moreover, even if no solitons are present, the asymptotic limit is not the (quasi-)periodic background! A precise description of the asymptotic limit in terms of Abelian integrals on the underlying Riemann surface is given in [9] (see [10] for a proof). In particular, the asymptotic limit can be split into parts, one which stems from the discrete spectrum (solitons) and one which stems from the continuous spectrum.

The soliton part can be understood by adding/removing the solitons using a Darboux-type transformation, that is, commutation methods for the underlying Jacobi operators. Hence the purpose of the present paper is to complement [10] and provide a detailed description of the double commutation method when applied to a short range perturbation of a quasi-periodic finite-gap solution of the Toda lattice. In particular, we are interested in the effect of one double commutation step on the scattering data.

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After introducing the Toda hierarchy in Section 2 and recalling some necessary facts on algebro-geometric quasi-periodic finite-gap solutions in Section 3 we briefly review the single and double commutation methods in Section 4 and compute the phase shift (in the Jacobian variety) caused by inserting one eigenvalue for both methods. In Section 5 we review direct scattering theory for Jacobi operators with different (quasi-)periodic asymptotics in the same isospectral class. As our main result we give a complete description of the effect of the double commutation method on the scattering data. In addition, we provide some detailed asymptotic formulas for the Jost functions $\psi_\pm(z,n)$ (which are normalized as $n\to\pm\infty$) at the *other* side, that is, as $n\to\mp\infty$. Our final Section 6 establishes the inverse scattering transform for this setting. Our main results here are the time dependence of both the scattering data and the kernel of the Gelfand-Levitan-Marchenko equation.

2. The Toda hierarchy

In this section we introduce the Toda hierarchy using the standard Lax formalism ([12]). We first review some basic facts from [1] (see also [21]).

We will only consider bounded solutions and hence require

Hypothesis H. 2.1. *Suppose $a(t)$, $b(t)$ satisfy*

$$a(t)\in\ell^\infty(\mathbb{Z},\mathbb{R}),\qquad b(t)\in\ell^\infty(\mathbb{Z},\mathbb{R}),\qquad a(n,t)\neq 0,\qquad (n,t)\in\mathbb{Z}\times\mathbb{R},$$

and let $t\mapsto(a(t),b(t))$ be differentiable in $\ell^\infty(\mathbb{Z})\oplus\ell^\infty(\mathbb{Z})$.

Associated with $a(t),b(t)$ is a Jacobi operator

$$H(t):\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z}),\qquad f\mapsto\tau(t)f, \tag{2.1}$$

where

$$\tau(t)f(n)=a(n,t)f(n+1)+a(n-1,t)f(n-1)+b(n,t)f(n) \tag{2.2}$$

and $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. Moreover, choose constants $c_0=1$, c_j, $1\le j\le r$, $c_{r+1}=0$, set

$$\begin{aligned} g_j(n,t) &= \sum_{\ell=0}^{j} c_{j-\ell}\langle\delta_n,H(t)^\ell\delta_n\rangle,\\ h_j(n,t) &= 2a(n,t)\sum_{\ell=0}^{j} c_{j-\ell}\langle\delta_{n+1},H(t)^\ell\delta_n\rangle+c_{j+1}, \end{aligned} \tag{2.3}$$

and consider the Lax operator

$$P_{2r+2}(t)=-H(t)^{r+1}+\sum_{j=0}^{r}(2a(t)g_j(t)S^+-h_j(t))H(t)^{r-j}+g_{r+1}(t), \tag{2.4}$$

where $S^\pm f(n)=f(n\pm1)$. Restricting to the two-dimensional nullspace $\mathrm{Ker}(\tau(t)-z)$, $z\in\mathbb{C}$, of $\tau(t)-z$, we have the following representation of $P_{2r+2}(t)$

$$P_{2r+2}(t)\Big|_{\mathrm{Ker}(\tau(t)-z)}=2a(t)G_r(z,t)S^+-H_{r+1}(z,t), \tag{2.5}$$

where $G_r(z,n,t)$ and $H_{r+1}(z,n,t)$ are monic polynomials in z of the type

$$\begin{aligned} G_r(z,n,t) &= \sum_{j=0}^{r} z^j g_{r-j}(n,t), \\ H_{r+1}(z,n,t) &= z^{r+1} + \sum_{j=0}^{r} z^j h_{r-j}(n,t) - g_{r+1}(n,t). \end{aligned} \tag{2.6}$$

A straightforward computation shows that the Lax equation

$$\frac{d}{dt}H(t) - [P_{2r+2}(t), H(t)] = 0, \qquad t \in \mathbb{R}, \tag{2.7}$$

is equivalent to

$$\begin{aligned} \mathrm{TL}_r(a(t),b(t))_1 &= \dot{a}(t) - a(t)\Big(g_{r+1}^+(t) - g_{r+1}(t)\Big) = 0, \\ \mathrm{TL}_r(a(t),b(t))_2 &= \dot{b}(t) - \Big(h_{r+1}(t) - h_{r+1}^-(t)\Big) = 0, \end{aligned} \tag{2.8}$$

where the dot denotes a derivative with respect to t and $f^{\pm}(n) = f(n\pm1)$. Varying $r \in \mathbb{N}_0$ yields the Toda hierarchy $\mathrm{TL}_r(a,b) = (\mathrm{TL}_r(a,b)_1, \mathrm{TL}_r(a,b)_2) = 0$. We will always consider r as a fixed, but arbitrary, value.

We recall that the Lax equation (2.7) implies existence of a unitary propagator $U_r(t,s)$ such that the family of operators $H(t)$, $t \in \mathbb{R}$, are unitarily equivalent, $H(t) = U_r(t,s)H(s)U_r(s,t)$. This also implies the basic existence and uniqueness theorem for the Toda hierarchy (see, e.g., [20], [19], or [21, Section 12.2]).

Theorem 2.2. *Suppose $(a_0,b_0) \in M = \ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$. Then there exists a unique integral curve $t \mapsto (a(t),b(t))$ in $C^\infty(\mathbb{R},M)$ of the Toda hierarchy, that is, $\mathrm{TL}_r(a(t),b(t)) = 0$, such that $(a(0),b(0)) = (a_0,b_0)$.*

Finally, we recall the following result from [3] (compare also [20]), which says that solutions which are asymptotically close to a background solution at the initial time stay close for all time.

Lemma 2.3. *Suppose $a(n,t)$, $b(n,t)$ and $\bar{a}(n,t)$, $\bar{b}(n,t)$ are two arbitrary bounded solutions of the Toda hierarchy satisfying (2.9) for one $t_0 \in \mathbb{R}$, then (2.9) holds for all $t \in \mathbb{R}$, that is,*

$$\sum_{n\in\mathbb{Z}} w(n)\Big(|a(n,t) - \bar{a}(n,t)| + |b(n,t) - \bar{b}(n,t)|\Big) < \infty, \tag{2.9}$$

where $w(n) > 0$.

3. Quasi-periodic finite-gap solutions

As a preparation for our next section we first need to recall some facts on quasi-periodic finite-gap solutions (again see [1] or [21]).

Let $\mathbb{M}$ be the Riemann surface associated with the following function

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1} \sqrt{z - E_j}, \qquad E_0 < E_1 < \cdots < E_{2g+1}, \tag{3.1}$$

where $g \in \mathbb{N}$ and $\sqrt{\cdot}$ is the standard root with branch cut along $(-\infty,0)$. $\mathbb{M}$ is a compact, hyperelliptic Riemann surface of genus g. A point on $\mathbb{M}$ is denoted by $p = (z, \pm R_{2g+2}^{1/2}(z)) = (z,\pm)$, $z \in \mathbb{C}$, or $p = \infty_\pm$, and the projection onto $\mathbb{C} \cup \{\infty\}$

by $\pi(p) = z$. The sets $\Pi_\pm = \{(z, \pm R_{2g+2}^{1/2}(z)) \mid z \in \mathbb{C}\setminus \bigcup_{j=0}^{g}[E_{2j}, E_{2j+1}]\} \subset \mathbb{M}$ are called upper, lower sheet, respectively.

Now pick g numbers (the Dirichlet eigenvalues)

$$(\hat\mu_j)_{j=1}^{g} = (\mu_j, \sigma_j)_{j=1}^{g} \tag{3.2}$$

whose projections lie in the spectral gaps, that is, $\mu_j \in [E_{2j-1}, E_{2j}]$. Associated with these numbers is the divisor $\mathcal{D}_{\underline{\hat\mu}}$ which is one at the points $\hat\mu_j$ and zero else. Using this divisor we introduce

$$\begin{aligned} \underline{z}(p,n,t) &= \underline{\hat A}_{p_0}(p) - \underline{\hat\alpha}_{p_0}(\mathcal{D}_{\underline{\hat\mu}}) - n\underline{\hat A}_{\infty_-}(\infty_+) + t\underline{U}_s - \underline{\hat\Xi}_{p_0} \in \mathbb{C}^g, \\ \underline{z}(n,t) &= \underline{z}(\infty_+, n, t), \end{aligned} \tag{3.3}$$

where $\underline{\Xi}_{p_0}$ is the vector of Riemann constants, $\underline{U}_s$ the b-periods of the Abelian differential Ω_s defined below, and $\underline{A}_{p_0}$ ($\underline{\alpha}_{p_0}$) is Abel's map (for divisors). The hat indicates that we regard it as a (single-valued) map from $\hat{\mathbb{M}}$ (the fundamental polygon associated with $\mathbb{M}$) to $\mathbb{C}^g$. We recall that the function $\theta(\underline{z}(p,n,t))$ has precisely g zeros $\hat\mu_j(n,t)$ (with $\hat\mu_j(0,0) = \hat\mu_j$), where $\theta(\underline{z})$ is the Riemann theta function of $\mathbb{M}$.

Taking a stationary solution of TL_g with constants c_j, $1 \le j \le g$, as initial condition for another equation $\widehat{\mathrm{TL}}_s$ with constants $\hat c_j$, $1 \le j \le s$, in the Toda hierarchy (2.8) one obtains the quasi-periodic finite gap solutions of the Toda hierarchy given by (see [21, Sections 13.1, 13.2])

$$\begin{aligned} a_q(n,t)^2 &= \tilde a^2 \frac{\theta(\underline{z}(n+1,t))\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n,t))^2}, \\ b_q(n,t) &= \tilde b + \sum_{j=1}^{g} c_j(g) \frac{\partial}{\partial w_j} \ln\Big(\frac{\theta(\underline{w} + \underline{z}(n,t))}{\theta(\underline{w} + \underline{z}(n-1,t))}\Big)\Big|_{\underline{w}=0}. \end{aligned} \tag{3.4}$$

The constants $\tilde a$, $\tilde b$, $c_j(g)$ depend only on the Riemann surface (see [21, Section 9.2]).

Introduce

$$\begin{aligned} \phi_q(p,n,t) &= C(n,t) \frac{\theta(\underline{z}(p,n+1,t))}{\theta(\underline{z}(p,n,t))} \exp\Big(\int_{p_0}^{p} \omega_{\infty_+,\infty_-}\Big), \\ \psi_q(p,n,t) &= C(n,0,t) \frac{\theta(\underline{z}(p,n,t))}{\theta(\underline{z}(p,0,0))} \exp\Big(n\int_{p_0}^{p} \omega_{\infty_+,\infty_-} + t\int_{p_0}^{p} \Omega_s\Big), \end{aligned} \tag{3.5}$$

where $C(n,t)$, $C(n,0,t)$ are real-valued,

$$C(n,t)^2 = \frac{\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n+1,t))}, \qquad C(n,0,t)^2 = \frac{\theta(\underline{z}(0,0))\theta(\underline{z}(-1,0))}{\theta(\underline{z}(n,t))\theta(\underline{z}(n-1,t))}, \tag{3.6}$$

and the sign of $C(n,t)$ is opposite to that of $a_q(n,t)$. $\omega_{\infty_+,\infty_-}$ is the Abelian differential of the third kind with poles at ∞_+ respectively ∞_- and Ω_s is an Abelian differential of the second kind with poles at ∞_+ respectively ∞_- whose Laurent expansion is given by the coefficients $(j+1)\hat c_{s-j}$ associated with $\widehat{\mathrm{TL}}_s$ (see [21, Sections 13.1, 13.2]). Then

$$\begin{aligned} \tau_q(t)\psi_q(p,n,t) &= \pi(p)\psi_q(p,n,t), \\ \frac{d}{dt}\psi_q(p,n,t) &= 2a_q(n,t)\hat G_s(p,n,t)\psi_q(p,n+1,t) - \hat H_{s+1}(p,n,t)\psi_q(p,n,t) \\ &= \hat P_{q,2s+2}(t)\psi_q(p,n,t), \end{aligned} \tag{3.7}$$

where we use the hat to distinguish the quantities associated with $\widehat{\mathrm{TL}}_s$ from those associated with TL_g.

The two branches $\psi_{q,\pm}(z,n,t) = \psi_q(p,n,t)$, $p = (z,\pm)$, of the Baker-Akhiezer function are linearly independent away from the branch points and their Wronskian is given by

$$W_q(\psi_{q,-}(z), \psi_{q,+}(z)) = \frac{R_{2g+2}^{1/2}(z)}{\prod_{j=1}^{g}(z-\mu_j)}.$$

Here $W_q(f,g) = a_q(n)(f(n)g(n+1) - f(n+1)g(n))$ is the usual modified Wronskian.

It is well known that the spectrum of $H_q(t)$ is time independent and consists of $g+1$ bands

$$\sigma(H_q) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]. \tag{3.8}$$

For further information and proofs we refer to [21, Chapter 9].

Finally, let us renormalize the Baker-Akhiezer function

$$\tilde{\psi}_q(p,n,t) = \frac{\psi_q(p,n,t)}{\psi_q(p,0,t)} \tag{3.9}$$

such that $\tilde{\psi}_q(p,0,t) = 1$ and let us define $\alpha_s(p,t)$ via

$$\exp\big(\alpha_s(p,t)\big) = \psi_q(p,0,t) = C(0,0,t)\frac{\theta(\underline{z}(p,0,t))}{\theta(\underline{z}(p,0,0))} \exp\Big(t\int_{p_0}^{p} \Omega_s\Big). \tag{3.10}$$

4. Commutation methods and N-soliton solutions

In this section we investigate commutation methods when applied to a quasi-periodic finite-gap background solution. In particular, we compute the phase shift (in the Jacobian variety) introduced by the solitons. This can be found for the case of one-dimensional Schrödinger operators in [7] (see also [6] for the elliptic case). The case of Jacobi operators seems to be missing and hence we provide the corresponding results to fill this gap. We want to be rather brief and refer to [8] or [21, Ch. 11] for further details in this connection. Since the time t does not play a role in this section, we will just omit it.

We start by inserting an eigenvalue using the single commutation method. Let H_q be a quasi-periodic finite-gap operator and let $\psi_{q,\pm}(z,n)$ be the branches of the Baker-Akhiezer function which are square summable near $\pm\infty$. Fix $\lambda_1 < \inf\sigma(H_q)$, $\sigma \in [-1,1]$, define

$$u_{q,\sigma}(\lambda_1,n) = \frac{1+\sigma}{2}\psi_{q,+}(\lambda_1,n) + \frac{1-\sigma}{2}\psi_{q,-}(\lambda_1,n), \tag{4.1}$$

and let $H_{q,\sigma}$ be the (self-adjoint) commuted operator associated with

$$(\tau_{q,\sigma}f)(n) = a_{q,\sigma}(n)f(n+1) + a_{q,\sigma}(n-1)f(n-1) + b_{q,\sigma}(n)f(n), \tag{4.2}$$

where (see [21, Sect. 11.2])

$$\begin{aligned} a_{q,\sigma}(n) &= -\frac{\sqrt{a_q(n)a_q(n+1)u_{q,\sigma}(\lambda_1,n)u_{q,\sigma}(\lambda_1,n+2)}}{u_{q,\sigma}(\lambda_1,n+1)}, \\ b_{q,\sigma}(n) &= \lambda_1 - a_q(n)\Big(\frac{u_{q,\sigma}(\lambda_1,n)}{u_{q,\sigma}(\lambda_1,n+1)} + \frac{u_{q,\sigma}(\lambda_1,n+1)}{u_{q,\sigma}(\lambda_1,n)}\Big) \\ &= b_q(n) + \partial^* \frac{a_q(n)u_{q,\sigma}(\lambda_1,n)}{u_{q,\sigma}(\lambda_1,n+1)}. \end{aligned} \tag{4.3}$$

$H_q - \lambda_1$ and $H_{q,\sigma} - \lambda_1$ restricted to the orthogonal complements of their corresponding one-dimensional null-spaces are unitarily equivalent and hence

$$\begin{aligned} \sigma_{ac}(H_{q,\sigma}) &= \sigma_{ac}(H_q), \qquad \sigma_{sc}(H_{q,\sigma}) = \sigma_{sc}(H_q) = \emptyset, \\ \sigma_{pp}(H_{q,\sigma}) &= \begin{cases} \{\lambda_1\}, & \sigma \in (-1,1) \\ \emptyset, & \sigma \in \{-1,1\} \end{cases}. \end{aligned}$$

Lemma 4.1. *Let H_q be a quasi-periodic finite-gap operator associated with the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and let $H_{q,\sigma}$, $-1 < \sigma < 1$, be the commuted operator associated with (4.2). Then we have*

$$a_{q,\sigma}(n) \sim a_{q,\pm 1}(n), \quad b_{q,\sigma}(n) \sim b_{q,\pm 1}(n) \qquad \text{as } n \to \pm\infty, \tag{4.4}$$

where $H_{q,\pm 1}$ are the quasi-periodic finite-gap operators associated with the Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}_{\pm 1}}$ defined via

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}_{\pm 1}}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) - \underline{A}_{p_0}(p_1) - \underline{A}_{p_0}(\infty_+), \qquad p_1 = (\lambda_1, \pm). \tag{4.5}$$

Proof. That $H_{q,\pm 1}$ is associated with the divisor $\mathcal{D}_{\underline{\hat{\mu}}_{\pm 1}}$ is shown in [21, Sect. 11.4] and the asymptotics follow since $u_{q,\sigma}(\lambda_1, n) \sim \frac{1\pm\sigma}{2} u_{q,\pm 1}(\lambda_1, n)$ as $n \to \mp\infty$. □

Similarly, one obtains the following result for the double commutation method. Let $\lambda_1 \in \mathbb{R}\backslash\sigma_{ess}(H_q)$, define (see [21, Sect. 11.6, (2.30)])

$$\begin{aligned} c_{q,\gamma}(\lambda_1,n) &= \frac{1}{\gamma} + \sum_{j=-\infty}^{n} \psi_{q,-}(\lambda_1,j)^2 \\ &= \frac{1}{\gamma} + W_{q,n}(\psi_{q,-}(\lambda_1), \dot{\psi}_{q,-}(\lambda_1)) = \frac{1}{\gamma} + \psi_{q,-}(\lambda_1,n)^2 \dot{\phi}_{q,-}(\lambda_1,n), \quad \gamma \neq 0, \end{aligned} \tag{4.6}$$

and let $H_{q,\gamma}$ be the doubly commuted operator associated with

$$\begin{aligned} a_{q,\gamma}(n) &= a_q(n)\frac{\sqrt{c_{q,\gamma}(\lambda_1,n-1)c_{q,\gamma}(\lambda_1,n+1)}}{c_{q,\gamma}(\lambda_1,n)}, \\ b_{q,\gamma}(n) &= b_q(n) - \partial^* \frac{a_q(n)\psi_{q,-}(\lambda_1,n)\psi_{q,-}(\lambda_1,n+1)}{c_{q,\gamma}(\lambda_1,n)}. \end{aligned} \tag{4.7}$$

Then

Lemma 4.2. *Let H_q be a quasi-periodic finite-gap operator associated with the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and let $H_{q,\gamma}$, $0 < \gamma < \infty$, be the doubly commuted operator*

associated with (4.7). Then we have

$$a_{q,\gamma}(n) = \begin{cases} a_q(n)(1+O(w(\lambda_1)^{2n}) & \text{as } n\to-\infty \\ a_{q,\infty}(n)(1+O(w(\lambda_1)^{-2n}) & \text{as } n\to+\infty \end{cases},$$

$$b_{q,\gamma}(n) = \begin{cases} b_q(n)(1+O(w(\lambda_1)^{2n}) & \text{as } n\to-\infty \\ b_{q,\infty}(n)(1+O(w(\lambda_1)^{-2n}) & \text{as } n\to+\infty \end{cases}, \tag{4.8}$$

where $w(z) = \exp(\int_{p_0}^{(z,+)} \omega_{\infty_+,\infty_-})$ *is the quasi-momentum map and* $H_{q,\infty}$ *is the quasi-periodic finite-gap operator associated with the Dirichlet divisor* $\mathcal{D}_{\underline{\hat{\mu}}_\infty}$ *defined via*

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}_{\lambda_1}}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + 2\underline{A}_{p_0}(\hat{\lambda}_1), \qquad \hat{\lambda}_1 = (\lambda_1, +). \tag{4.9}$$

Proof. Since the double commutation method can be obtained via two single commutation steps (see [21, Sect. 11.5]), the result is a consequence of our previous lemma. The asymptotics follow from (4.7) and (3.5). □

Clearly, if we add k eigenvalues $\lambda_1, \dots, \lambda_k$, then the asymptotics at $+\infty$ are given by the quasi periodic operator associated with the Dirichlet divisor

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}_{\lambda_1,\dots,\lambda_k}}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + 2\sum_{j=1}^{k}\underline{A}_{p_0}(\hat{\lambda}_j), \qquad \hat{\lambda}_j = (\lambda_j, +). \tag{4.10}$$

In particular, by choosing at least one eigenvalue in each gap, we can attain any prescribed asymptotics in the given isospectral class by Lemma 9.1 in [21].

Remark 4.3. *If* $a_q(n,t)$, $b_q(n,t)$ *is a quasi-periodic solution of the Toda hierarchy and* $\psi_q(p,n,t)$ *is the corresponding time dependent Baker-Akhiezer function, then* $a_{q,\gamma}(n,t)$, $b_{q,\gamma}(n,t)$ *is a solution of the Toda hierarchy which is centered at*

$$2\alpha(\lambda_1)(n - v(\lambda_1)t) + \ln(\gamma) = 0, \tag{4.11}$$

where

$$\alpha(\lambda_1) = \mathrm{Re}\int_{p_0}^{(\lambda_1,-)} \omega_{\infty_+,\infty_-}, \qquad v(\lambda_1) = -\frac{1}{\alpha(\lambda_1)}\mathrm{Re}\int_{p_0}^{(\lambda_1,-)} \Omega_s. \tag{4.12}$$

5. Scattering theory

In this section we review scattering theory for Jacobi operators with steplike quasi-periodic finite-gap background in the same isospectral class following [4]. Our only new result in this section will be a full description of the effect of the double commutation method on the scattering data in Lemma 5.5.

More precisely, we will take two quasi-periodic finite-gap operators $H_q^\pm$ associated with the sequences $a_q^\pm$, $b_q^\pm$ in the same isospectral class,

$$\sigma(H_q^+) = \sigma(H_q^-) \equiv \Sigma = \bigcup_{j=0}^{g}[E_{2j}, E_{2j+1}], \tag{5.1}$$

but with possibly different Dirichlet data $\{\hat{\mu}_j^\pm\}_{j=1}^g$. We will add $\pm$ as a superscript to all data introduced in Section 3 to distinguish between the corresponding data of H_q^+ and H_q^-. To avoid excessive sub/superscripts we abbreviate

$$\psi_q^\pm(z,n) = \psi_{q,\pm}^\pm(z,n) \quad \text{and} \quad \breve{\psi}_q^\pm(z,n) = \psi_{q,\mp}^\pm(z,n), \tag{5.2}$$

that is, $\psi_q^{\pm}(z,n)$ is the solution of $H_q^{\pm}$ decaying near $\pm\infty$ and $\breve{\psi}_q^{\pm}(z,n)$ is the solution of $H_q^{\pm}$ decaying near $\mp\infty$. Note that for $\lambda \in \Sigma$ we have $\breve{\psi}_q^{\pm}(\lambda,n) = \overline{\psi_q^{\pm}(\lambda,n)}$.

Let $a(n)$, $b(n)$ be sequences satisfying

$$\sum_{n=0}^{\pm\infty} |n| \Big(|a(n) - a_q^{\pm}(n)| + |b(n) - b_q^{\pm}(n)| \Big) < \infty \tag{5.3}$$

and denote the corresponding operator by H.

Theorem 5.1. *Assume (5.3). Then there exist solutions $\psi_{\pm}(z,.)$, $z \in \mathbb{C}$, of $\tau\psi = z\psi$ satisfying*

$$\lim_{n\to\pm\infty} |w(z)^{\mp n}(\psi_{\pm}(z,n) - \psi_q^{\pm}(z,n))| = 0, \tag{5.4}$$

where $w(z) = \exp(\int_{p_0}^{(z,+)} \omega_{\infty_+,\infty_-})$ is the quasi-momentum map.

Theorem 5.2. *Assume (5.3). Then we have $\sigma_{ess}(H) = \Sigma$, the point spectrum of H is finite and confined to the spectral gaps of $H_q^{\pm}$, that is, $\sigma_p(H) = \{\rho_j\}_{j=1}^q \subset \mathbb{R}\backslash\Sigma$. Furthermore, the essential spectrum of H is purely absolutely continuous.*

Using the fact that $\psi_q^{\pm}(p,n)$ form an orthonormal basis for $L^2(\partial\Pi_+, d\omega^{\pm})$, where

$$d\omega^{\pm} = \frac{\prod_{j=1}^g (\pi - \mu_j^{\pm})}{R_{2g+2}^{1/2}} d\pi, \tag{5.5}$$

we can define

$$K_{\pm}(n,m) = 2\mathrm{Re} \int_{\Sigma} \psi_{\pm}(\lambda,n)\psi_q^{\pm}(\lambda,m) d\omega^{\pm} \tag{5.6}$$

implying

$$\psi_{\pm}(z,n) = \sum_{m=n}^{\pm\infty} K_{\pm}(n,m)\psi_q^{\pm}(z,m). \tag{5.7}$$

Next we define the coefficients of the scattering matrix via the scattering relations

$$\psi_{\mp}(\lambda,n) = \alpha_{\pm}(\lambda)\overline{\psi_{\pm}(\lambda,n)} + \beta_{\pm}(\lambda)\psi_{\pm}(\lambda,n), \qquad \lambda \in \Sigma, \tag{5.8}$$

where

$$\begin{aligned} \alpha_{\pm}(\lambda) &= \frac{W(\psi_{\pm}(\lambda), \psi_{\mp}(\lambda))}{W(\psi_{\pm}(\lambda), \overline{\psi_{\pm}(\lambda)})} = \frac{\prod_{j=1}^g (\lambda - \mu_j^{\pm})}{R_{2g+2}^{1/2}(\lambda)} W(\psi_-(\lambda), \psi_+(\lambda)), \\ \beta_{\pm}(\lambda) &= \frac{W(\psi_{\mp}(\lambda), \overline{\psi_{\pm}(\lambda)})}{W(\psi_{\pm}(\lambda), \overline{\psi_{\pm}(\lambda)})} = \mp \frac{\prod_{j=1}^g (z - \mu_j^{\pm})}{R_{2g+2}^{1/2}(z)} W(\psi_{\mp}(\lambda), \overline{\psi_{\pm}(\lambda)}), \end{aligned} \tag{5.9}$$

and $W_n(f,g) = a(n)(f(n)g(n+1) - f(n+1)g(n))$ denotes the Wronskian. Transmission $T_{\pm}(\lambda)$ and reflection $R_{\pm}(\lambda)$ coefficients are then defined by

$$T_{\pm}(\lambda) = \alpha_{\pm}^{-1}(\lambda), \qquad R_{\pm}(\lambda) = \frac{\beta_{\pm}(\lambda)}{\alpha_{\pm}(\lambda)} = \frac{W(\psi_{\mp}(\lambda), \overline{\psi_{\pm}(\lambda)})}{W(\psi_{\pm}(\lambda), \psi_{\mp}(\lambda))}. \tag{5.10}$$

The norming constants $\gamma_{\pm,j}$ corresponding to $\rho_j \in \sigma_p(H)$ are given by

$$\gamma_{\pm,j}^{-1} = \sum_{n\in\mathbb{Z}} |\psi_{\pm}(\rho_j,n)|^2, \qquad 1 \le j \le q. \tag{5.11}$$

Note that $\gamma_{\pm,j} = 0$ if ρ_j coincides with a pole $\hat{\mu}_\ell^\pm \in \Pi_\pm$ of $\psi_\pm(z,.)$. To avoid this, one could remove the poles by introducing $\hat{\psi}_\pm(z,.)$ as we did in [4]. Since this normalization cancels out in the Gel'fand-Levitan-Marchenko equation and unnecessarily complicates the formulas below, we will allow zero norming constants.

Moreover, $\psi_\pm(\rho_j,.) = c_j^\pm \psi_\mp(\rho_j,.)$ with $c_j^+ c_j^- = 1$.

Lemma 5.3. *The coefficients $T_\pm(\lambda)$, $R_\pm(\lambda)$ are bounded for $\lambda \in \Sigma$, continuous for $\lambda \in \Sigma$ except at possibly the band edges E_j, and fulfill*

$$T_+(\lambda)\overline{T_-(\lambda)} + |R_\pm(\lambda)|^2 = 1, \qquad \lambda \in \Sigma, \tag{5.12}$$

$$T_\pm(\lambda)\overline{R_\pm(\lambda)} + \overline{T_\pm(\lambda)}R_\mp(\lambda) = 0, \qquad \lambda \in \Sigma. \tag{5.13}$$

In particular,

$$|T_\pm(\lambda)|^2 \prod_{j=1}^{g} \frac{\lambda - \mu_j^\pm}{\lambda - \mu_j^\mp} + |R_\pm(\lambda)|^2 = 1, \tag{5.14}$$

and hence $|R_\pm(\lambda)|^2 \le 1$ with equality only possibly at the band edges $\{E_j\}$. The transmission coefficients $T_\pm(\lambda)$ have a meromorphic continuation to $\mathbb{C}\backslash\Sigma$ with simple poles at $\mu_j^\pm$ if $\hat{\mu}_j^\pm \in \Pi_\mp$ and simple poles at ρ_j,

$$(\mathrm{Res}_{\rho_j} T_\pm(\lambda))^2 = \gamma_{+,j}\gamma_{-,j} \frac{R_{2g+2}(\rho_j)}{\prod_{l=1}^{g}(\rho_j - \mu_l^\pm)^2}. \tag{5.15}$$

In addition, $T_\pm(z) \in \mathbb{R}$ as $z \in \mathbb{R}\backslash\Sigma$ and

$$T_\pm(\infty) = \prod_{n=-\infty}^{-1} \frac{a(n)}{a_q^-(n)} \prod_{n=0}^{\infty} \frac{a(n)}{a_q^+(n)}. \tag{5.16}$$

The sets

$$S_\pm(H) = \{R_\pm(\lambda), \lambda \in \Sigma;\ (\rho_j, \gamma_{\pm,j}), 1 \le j \le q\} \tag{5.17}$$

are called left/right scattering data for H.

Theorem 5.4. *The kernel $K_\pm(n,m)$ of the transformation operator satisfies the Gel'fand-Levitan-Marchenko equation*

$$K_\pm(n,m) + \sum_{l=n}^{\pm\infty} K_\pm(n,l)F^\pm(l,m) = \frac{\delta(n,m)}{K_\pm(n,n)}, \qquad \pm m \ge \pm n, \tag{5.18}$$

where

(5.19)
$$F^\pm(m,n) = 2\mathrm{Re} \int_\Sigma R_\pm(\lambda)\psi_q^\pm(\lambda,m)\psi_q^\pm(\lambda,n)d\omega^\pm + \sum_{j=1}^{q} \gamma_{\pm,j}\psi_q^\pm(\rho_j,n)\psi_q^\pm(\rho_j,m).$$

Note that the apparent poles $\mu_\ell^\pm$ cancel with the zeros of $d\omega^\pm$ and $\gamma_{\pm,j}$ at these points.

The operator H can be uniquely reconstructed from $S_\pm(H)$ by solving the Gel'fand-Levitan-Marchenko equation. We refer to [4] for further details.

Finally, we come to our principal new result in this section and investigate the connection with the double commutation method. The scattering data of the operators H, H_γ are related as follows.

Lemma 5.5. *Let H be a given Jacobi operator satisfying (5.3) and choose $\rho_{q+1} \in \mathbb{R}\backslash\sigma(H)$, $\gamma > 0$. Then the doubly commuted operator H_γ, $\gamma > 0$, defined via $\psi_-(\rho_{q+1})$ as in Section 4, satisfies*

$$a_\gamma(n) \sim \left\{ \begin{array}{ll} a_q^-(n) & \text{as } n \to -\infty \\ a_q^\infty(n) & \text{as } n \to +\infty \end{array} \right., \qquad b_\gamma(n) \sim \left\{ \begin{array}{ll} b_q^-(n) & \text{as } n \to -\infty \\ b_q^\infty(n) & \text{as } n \to +\infty \end{array} \right., \tag{5.20}$$

such that (5.3) still holds, where H_q^∞ is associated with Dirichlet data given by

$$\underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\mu}^\infty}) = \underline{\alpha}_{p_0}(\mathcal{D}_{\underline{\mu}^+}) + 2\underline{A}_{p_0}(\hat{\rho}_{q+1}). \tag{5.21}$$

It has the scattering data

$$R_{-,\gamma}(\lambda) = R_-(\lambda), \tag{5.22}$$

$$R_{+,\gamma}(\lambda) = \frac{\theta(\underline{z}^\infty(\hat{\lambda},0))}{\theta(\underline{z}^+(\hat{\lambda},0))} \frac{\theta(\underline{z}^+(\hat{\lambda}^*,0))}{\theta(\underline{z}^\infty(\hat{\lambda}^*,0))} B(\lambda,\rho_{q+1})^2 R_+(\lambda), \tag{5.23}$$

$$T_{+,\gamma}(z) = C\, \frac{\theta(\underline{z}^+(\hat{z}^*,0))}{\theta(\underline{z}^\infty(\hat{z}^*,0))} B(z,\rho_{q+1}) T_+(z), \tag{5.24}$$

$$T_{-,\gamma}(z) = \frac{1}{C}\, \frac{\theta(\underline{z}^\infty(\hat{z},0))}{\theta(\underline{z}^+(\hat{z},0))} B(z,\rho_{q+1}) T_-(z), \tag{5.25}$$

where

$$B(z,\rho) = \exp\Big(- \int_{E(\rho)}^{(\rho,+)} \omega_{\hat{z}\hat{z}^*} \Big) \tag{5.26}$$

is the Blaschke factor and $\hat{\lambda} = (\lambda,+)$, $\hat{z} = (z,+)$. The constant C is given by

$$C = \sqrt{\frac{\theta(\underline{z}^\infty(\infty_+,0))\theta(\underline{z}^\infty(\infty_-,0))}{\theta(\underline{z}^+(\infty_+,0))\theta(\underline{z}^+(\infty_-,0))}} > 0. \tag{5.27}$$

The norming constants $\gamma_{-,j}$ corresponding to $\rho_j \in \sigma_p(H)$, $j = 1, \dots, q$, (cf. (5.11)) remain unchanged except for an additional eigenvalue ρ_{q+1} with norming constant $\gamma_{-,q+1} = \gamma$. The norming constants $\gamma_{+,j,\gamma}$, $j = 1, \dots, q$, are given by

$$\gamma_{+,j,\gamma} = \frac{1}{C^2} \left(\frac{\theta(\underline{z}^\infty(\hat{\rho}_j,0))}{\theta(\underline{z}^+(\hat{\rho}_j,0))} \right)^2 B(\rho_j,\rho_{q+1})^2 \gamma_{+,j}, \qquad \hat{\rho}_j = (\rho_j,+), \tag{5.28}$$

and the additional norming constant $\gamma_{+,q+1,\gamma}$ reads

$$\gamma_{+,q+1,\gamma} = C^2 \frac{\prod_{j=1}^g (\rho_{q+1} - \mu_j^+)^2}{\gamma R_{2g+2}(\rho_{q+1})} \Big(\frac{\theta(\underline{z}^+(\hat{\rho}_{q+1}^*,0))}{\theta(\underline{z}^\infty(\hat{\rho}_{q+1}^*,0))} T_+(\rho_{q+1}) \mathrm{Res}_{z=\rho_{q+1}} B(z,\rho_{q+1}) \Big)^2. \tag{5.29}$$

Proof. First we show that (5.3) still holds. Note that

$$a_\gamma(n) = \left\{ \begin{array}{ll} a(n)(1 + O(w(\rho_{q+1})^{2n}) & \text{as } n \to -\infty \\ a_\infty(n)(1 + O(w(\rho_{q+1})^{-2n}) & \text{as } n \to +\infty \end{array} \right.,$$

$$b_\gamma(n) = \left\{ \begin{array}{ll} b(n)(1 + O(w(\rho_{q+1})^{2n}) & \text{as } n \to -\infty \\ b_\infty(n)(1 + O(w(\rho_{q+1})^{-2n}) & \text{as } n \to +\infty \end{array} \right..$$

Hence the asymptotics near $-\infty$ are clearly unchanged and for $+\infty$ it suffices to check $\gamma = \infty$. By Lemma 5.7 below,

$$\frac{c_\infty(\rho_{q+1},n+1)}{c_\infty(\rho_{q+1},n)} = \frac{c_\infty(\rho_{q+1},n+1)\psi_+(\rho_{q+1},n+1)}{c_\infty(\rho_{q+1},n)\psi_+(\rho_{q+1},n+1)} = \frac{c_{q,\infty}^+(\rho_{q+1},n+1)}{c_{q,\infty}^+(\rho_{q+1},n)}(1 + C(n)),$$

where $\sum_{n\in\mathbb{N}} n|C(n)| < \infty$. Therefore

$$a_\infty(n) = a(n)\frac{\sqrt{c_\infty(\rho_{q+1}, n-1)c_\infty(\rho_{q+1}, n+1)}}{c_\infty(\rho_{q+1}, n)} \to a_{q,\infty}(n) \tag{5.30}$$

such that $\sum_{n\in\mathbb{N}} n|a_\infty(n) - a_{q,\infty}(n)| < \infty$ and similarly for $b_\infty(n)$.

Now we turn to the scattering data. By [21, Lemma 11.19], the Jost solutions $\psi_{\pm,\gamma}(z,n)$ of H_γ are up to a constant given by

(5.31)
$$u_{\pm,\gamma}(z,n) = \frac{c_\gamma(\rho_{q+1}, n)\psi_\pm(z,n) - \frac{1}{z-\rho_{q+1}}\psi_-(\rho_{q+1}, n)W_{n-1}(\psi_-(\rho_{q+1}), \psi_\pm(z))}{\sqrt{c_\gamma(\rho_{q+1}, n-1)c_\gamma(\rho_{q+1}, n)}}.$$

Since this constant is equal to 1 for $\psi_{-,\gamma}(z,n)$ the fact that R_- is unchanged follows from its definition and [21, (11.107)]. The transmission coefficients are reconstructed from $R_-(\lambda)$ using [4, Theorem 3.6] and $R_{+,\gamma}(\lambda)$ follows then from

$$R_{+,\gamma}(\lambda) = -\frac{\overline{T_{-,\gamma}(\lambda)}}{T_{-,\gamma}(\lambda)} R_{-,\gamma}(\lambda).$$

That the norming constants $\gamma_{-,j}$ are unchanged follows from [21, Lemma 11.14]. For $\gamma_{+,j,\gamma}$, $j = 1, \dots, q+1$, we use (5.15)

$$\gamma_{+,j,\gamma} = \frac{\prod_{l=1}^g (\rho_j - \mu_l^\pm)^2 (\mathrm{Res}_{\rho_j} T_{\pm,\gamma}(\lambda))^2}{R_{2g+2}(\rho_j)\gamma_{-,j}}.$$

□

Remark 5.6. *If we choose $\rho = \rho_j \in \sigma_p(H)$ and $\gamma = -\gamma_{-,j}$, then the eigenvalue ρ_j is removed from the spectrum and it is straightforward to see that an analogous result holds.*

The following result used in the previous proof is of independent interest.

Lemma 5.7. *Let H be a given Jacobi operator satisfying (5.3). Then for every $z \in \mathbb{C}\backslash\Sigma$ and every $k \in \mathbb{N}$ we have*

$$\psi_\mp(z,n)\psi_\pm(z,n+k) - \alpha_\pm(z)\psi_q^\pm(z,n+k)\bar{\psi}_q^\pm(z,n) = (k+1)|w(z)|^k C_\pm(z,n), \tag{5.32}$$

where $\sum_{\pm n\in\mathbb{N}} n|C_\pm(z,n)| < \infty$.

Similarly, one has

$$c_\infty(z,n)\psi_\pm(z,n+k) - \alpha_\pm(z)c_{q,\infty}^\pm(z,n)\psi_q^\pm(z,n+k) = \tilde{C}_\pm(z,n,k), \tag{5.33}$$

where $\sum_{\pm n\in\mathbb{N}} n|\tilde{C}_\pm(z,n,k)| < \infty$ and $c_\infty(z,n)$, $c_{q,\infty}^\pm(z,n)$ are defined as in (4.6).

Proof. The proof is an extension of [14, Lemma 3.4] and we will only consider the '+' case. First recall that the Green's functions of H_q^+ and H are given by

$$G_q^+(z,n,m) = \frac{\bar{\psi}_q^+(z,m)\psi_q^+(z,n)}{W_q(\bar{\psi}_q^+(z), \psi_q^+(z))}, \quad G(z,n,m) = \frac{\psi_-(z,m)\psi_+(z,n)}{W(\psi_-(z), \psi_+(z))}, \quad n \ge m,$$

respectively. Considering matrix elements in the second resolvent identity $(H - z)^{-1} - (H_q^+ - z)^{-1} = (H-z)^{-1}(H_q^+ - H)(H_q^+ - z)^{-1}$ we obtain

$$C_+(z,n) = \frac{|w(z)|^{-k}}{k+1}\sum_{m\in\mathbb{Z}} W(\psi_-(z), \psi_+(z))G(z,n,m)(H_q^+ - H)G_q^+(z,m,n+k)$$

for $z \in \mathbb{C}\backslash\sigma(H)$. Since the poles of $W(\psi_-(z), \psi_+(z))G(z,n,m)$ at $z \in \sigma_p(H)$ are removable, the formula holds for all $z \in \mathbb{C}\backslash\Sigma$. Estimating the right hand side using $|G(z,n,m)| \leq const\,|w(z)|^{|n-m|}$ and $|G_q^+(z,n+k,m)| \leq const\,|w(z)|^{|n+k-m|}$ we obtain

$$|C_+(z,n)| \leq \frac{C|w(z)|^{-k}}{k+1} \sum_{m\in\mathbb{Z}} |w(z)|^{|n-m|+|n-m+k|}(2|a(m)-a_q^+(m)|+|b(m)-b_q^+(m)|).$$

We split the sum into three parts

$$|w(z)|^{|n-m|+|n-m+k|} = \begin{cases} |w(z)|^k |w(z)|^{2|n-m|} & m < n, \\ |w(z)|^k & n \leq m \leq n+k, \\ |w(z)|^k |w(z)|^{2|n-m+k|} & m > n+k. \end{cases}$$

Since (5.3) holds for $c(m) = 2|a(m)-a_q^+(m)|+|b(m)-b_q^+(m)|$ and we have $|w(z)| < 1$ for $z \in \mathbb{C}\backslash\Sigma$, we can apply Lemma A.1 to verify $\sum_{n\in\mathbb{N}} n|C_+(z,n)| < \infty$.

The claim (5.33) is a consequence of (4.6) and (5.32)

$$\begin{aligned} c_\infty(z,n)\psi_+(z,n+k)^2 &= \sum_{j=-\infty}^{n} \psi_-(z,j)^2 \psi_+(z,n+k)^2 \\ &= \alpha_+(z)^2 c_{q,\infty}^+(z,n)\psi_q^+(z,n+k)^2 + \tilde{C}_\pm(z,n,k)\psi_q^+(z,n+k), \end{aligned}$$

where $\tilde{C}_\pm(z,n,k)$ again can be estimated using Lemma A.1. □

6. Inverse scattering transform

Let $a(n,t)$, $b(n,t)$ be a solution of the Toda hierarchy satisfying

$$\sum_{n=0}^{\pm\infty} |n|\Big(|a(n,t) - a_q^\pm(n,t)| + |b(n,t) - b_q^\pm(n,t)|\Big) < \infty. \tag{6.1}$$

Note that by Lemma 2.3 it suffices to check (6.1) for one $t_0 \in \mathbb{R}$ (as background take $H_q^-(t)$ and insert g eigenvalues such that the asymptotics on the other side are given by $H_q^+(t)$).

Jost solutions, transmission and reflection coefficients depend now on an additional parameter $t \in \mathbb{R}$. The Jost solutions $\psi_\pm(z,n,t)$ are normalized such that

$$\tilde{\psi}_\pm(z,n,t) = \tilde{\psi}_q^\pm(z,n,t)\,(1+o(1)) \quad \text{as } n \to \pm\infty,$$

where (c.f. (3.9))

$$\tilde{\psi}_q^\pm(z,n,t) = \frac{\psi_q^\pm(z,n,t)}{\psi_q^\pm(z,0,t)} =: \exp(-\alpha_s^\pm(z,t))\psi_q^\pm(z,n,t). \tag{6.2}$$

Moreover, we set

$$\exp(\bar{\alpha}_s^\pm(z,t)) = \bar{\psi}_q^\pm(z,0,t). \tag{6.3}$$

Note that we have $\overline{\exp(\alpha_s^\pm(z,t))} = \exp(\bar{\alpha}_s^\pm(z,t))$ for $\lambda \in \Sigma$.

Transmission and reflection coefficients are then defined via the normalized Jost solutions $\tilde{\psi}_\pm(z,n,t)$. Moreover,

$$\sigma(H(t)) \equiv \sigma(H), \qquad t \in \mathbb{R}. \tag{6.4}$$

To avoid the poles of the Baker-Akhiezer function, we will assume that none of the eigenvalues ρ_j coincides with a Dirichlet eigenvalue $\mu_k^\pm(0,0)$. This can be done without loss of generality by shifting the initial time $t_0 = 0$ if necessary.

Remark 6.1. *Due to this assumption there is no need to remove these poles for the definition of $\gamma_{\pm,j}$, as we did in* [2], [4]. *Since the Dirichlet eigenvalues rotate in their gap, the factor needed to remove the poles would only unnecessarily complicate the time evolution of the norming constants. Moreover, these factors would eventually cancel in the Gel'fand-Levitan-Marchenko equation, which is the only interesting object from the inverse spectral point of view in the first place.*

Lemma 6.2. *Let $(a(t), b(t))$ be a solution of the Toda hierarchy such that (5.3) holds. The functions*

$$\psi_\pm(z,n,t) = \exp(\alpha_s^\pm(z,t))\tilde{\psi}_\pm(z,n,t) \tag{6.5}$$

satisfy

$$H(t)\psi_\pm(z,n,t) = z\psi_\pm(z,n,t), \qquad \frac{d}{dt}\psi_\pm(z,n,t) = \hat{P}_{2s+2}(t)\psi_\pm(z,n,t). \tag{6.6}$$

Proof. We proceed as in [3], [20, Theorem 3.2]. The Jost solutions $\psi_\pm(z,n,t)$ are continuously differentiable with respect to t by the same arguments as for z (compare [2, Theorem 4.2]) and the derivatives are equal to the derivatives of the Baker-Akhiezer functions as $n \to \pm\infty$.

For $z \in \rho(H)$, the solution $u_\pm(z,n,t)$ of (6.6) with initial condition $\psi_\pm(z,n,0) \in \ell^2_\pm(\mathbb{Z})$ remains square summable near $\pm\infty$ for all $t \in \mathbb{R}$ (see [19] or [21, Lemma 12.16]), that is, $u_\pm(z,n,t) = C_\pm(t)\psi_\pm(z,n,t)$. Letting $n \to \pm\infty$ we see $C_\pm(t) = 1$. The general result for all $z \in \mathbb{C}$ now follows from continuity. $\square$

This implies

Theorem 6.3. *Let $(a(t), b(t))$ be a solution of the Toda hierarchy such that (5.3) holds. The time evolution for the scattering data is given by*

$$T_\pm(z,t) = T_\pm(z,0)\exp(\alpha_s^\mp(z,t) - \bar{\alpha}_s^\pm(z,t)), \tag{6.7}$$

$$R_\pm(\lambda,t) = R_\pm(\lambda,0)\exp(\alpha_s^\pm(\lambda,t) - \bar{\alpha}_s^\pm(\lambda,t)), \quad \lambda \in \Sigma, \tag{6.8}$$

$$\gamma_{\pm,j}(t) = \gamma_{\pm,j}(0)\exp(2\alpha_s^\pm(\rho_j,t)), \qquad 1 \le j \le q. \tag{6.9}$$

Proof. Since the Wronskian of two solutions satisfying (6.6) does not depend on n or t (see [19], [21, Lemma 12.15]), we have

$$\begin{aligned} T_\pm(z,t) &= \frac{W(\overline{\tilde{\psi}_\pm(z,t)}, \tilde{\psi}_\pm(z,t))}{W(\tilde{\psi}_\mp(z,t), \tilde{\psi}_\pm(z,t))} = \frac{\exp(\alpha_s^\mp(z,t) + \alpha_s^\pm(z,t))}{\exp(\bar{\alpha}_s^\pm(z,t) + \alpha_s^\pm(z,t))} \frac{W(\overline{\psi_\pm(z,t)}, \psi_\pm(z,t))}{W(\psi_\mp(z,t), \psi_\pm(z,t))} \\ &= \exp(\alpha_s^\mp(z,t) - \bar{\alpha}_s^\pm(z,t))T_\pm(z,0). \end{aligned}$$

The result for $R_\pm(\lambda,t)$ follows similarly. The time dependence of $\gamma_{\pm,j}(t)$ follows from $\|\psi_\pm(\rho_j,.,t)\| = \|\hat{U}_s(t,0)\psi_\pm(\rho_j,.,0)\| = \|\psi_\pm(\rho_j,.,0)\|$. $\square$

Corollary 6.4. *The quantity $T_\pm(\lambda,t)\overline{T_\mp(\lambda,t)} = 1 - |R_\pm(\lambda,t)|^2$, $\lambda \in \Sigma$, does not depend on t.*

Another straightforward consequence is:

Theorem 6.5. *The time dependence of the kernel of the Gel'fand-Levitan-Marchenko equation is given by*

$$F^{\pm}(m,n,t) = 2\mathrm{Re}\int_{\Sigma} R_{\pm}(\lambda,0)\psi_q^{\pm}(\lambda,m,t)\psi_q^{\pm}(\lambda,n,t)d\omega^{\pm}(0) + \sum_{j=1}^{q}\gamma_{\pm,j}(0)\psi_q^{\pm}(\rho_j,m,t)\psi_q^{\pm}(\rho_j,n,t).$$

Proof. Just employ Theorem 6.3 to rewrite

$$F^{\pm}(m,n,t) = \int_{\partial\Pi_+} R_{\pm}(p,t)\tilde{\psi}_q^{\pm}(p,m,t)\tilde{\psi}_q^{\pm}(p,n,t)d\omega^{\pm}(t) + \sum_{j=1}^{q}\gamma_{\pm,j}(t)\tilde{\psi}_q^{\pm}(\rho_j,n,t)\tilde{\psi}_q^{\pm}(\rho_j,m,t),$$

where we also use that $\exp(\alpha_s^{\pm}(\lambda,t)+\bar{\alpha}_s^{\pm}(\lambda,t)) = \prod_{j=1}^{g}\frac{\lambda-\mu_j^{\pm}(0,t)}{\lambda-\mu_j^{\pm}(0,0)}$. □

Finally we note ([19], [21, Section 14.5])

Lemma 6.6. *Let $(a(t),b(t))$ be a solution of the Toda hierarchy such that (5.3) holds. Choose $\rho\in\mathbb{R}\backslash\sigma(H)$ and $\gamma>0$. Then $(a_\gamma(t),b_\gamma(t))$ defined via $\psi_-(\rho,n,t)$ using the double commutation method is again a solution of the Toda hierarchy such that (5.3), with $H_q^+(t)$ accordingly changed, holds.*

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Appendix A. Some estimates

In the proof of Lemma 5.7 we need the following elementary result. Consider a sequence $c\in\ell(\mathbb{Z})$ and abbreviate

$$\|c\|_\infty = \sup_{n\in\mathbb{Z}}|c(n)|, \quad \|c\|_1 = \sum_{n=0}^{\infty}|c(n)|, \quad \|c\|_{1,1} = \sum_{n=1}^{\infty}n|c(n)|. \tag{A.1}$$

Lemma A.1. *Suppose w is some complex number with $|w|<1$ and $c\in\ell(\mathbb{Z})$ satisfies $\|c\|_\infty, \|c\|_{1,1}<\infty$.*

Then

$$\|\sum_{m=0}^{\infty}c(n+m)w^m\|_{1,1} \le \frac{1}{1-|w|}\|c\|_{1,1}$$

and

$$\|\sum_{m=0}^{\infty}c(n-m)w^m\|_{1,1} \le \frac{1}{1-|w|}\|c\|_{1,1} + \frac{|w|}{(1-|w|)^2}\|c\|_1 + \frac{|w|^2}{(1-|w|)^3}\|c\|_\infty.$$

Proof. The first estimate follows from

$$\begin{aligned}
\| \sum_{m=0}^{\infty} c(n+m) w^m \|_{1,1} &= \sum_{n=1}^{\infty} n \Big| \sum_{m=0}^{\infty} c(n+m) w^m \Big| \\
&\le \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n+m) |c(n+m)| |w|^m \\
&= \sum_{m=0}^{\infty} \|c\|_{1,1} |w|^m = \frac{1}{1-|w|} \|c\|_{1,1}.
\end{aligned}$$

Similarly, the second follows from

$$\begin{aligned}
\| \sum_{m=0}^{\infty} c(n-m) w^m \|_{1,1} &= \sum_{n=0}^{\infty} n \Big| \sum_{m=0}^{\infty} c(n-m) w^m \Big| \\
&\le \sum_{m=0}^{\infty} \left(\frac{m(m-1)}{2} \|c\|_{\infty} + m \|c\|_1 + \|c\|_{1,1} \right) |w|^m.
\end{aligned}$$

□

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